Lecture 1: Representations and Characters of Finite Groups

Goal: Establish the classical theory of group representations over $\mathbb{C}$, develop character theory, and lay the foundation for modular representations.

1. Representations and Characters

Definition 1.1 (Representation). Let G be a finite group and V a finite-dimensional vector space over a field F. A *representation* of G over F is a group homomorphism

$$\rho : G \to \mathrm{GL}(V).$$

Definition 1.2 (Degree). The degree of ρ is $\dim_F(V)$, often denoted $\deg(\rho)$.

Definition 1.3 (Character). The *character* of ρ, denoted χ_ρ, is the function

$$\chi_\rho(g) := \mathrm{Tr}(\rho(g)), \quad g \in G.$$

Lemma 1.4 (Conjugacy Class Function). Characters are constant on conjugacy classes:

$$\chi(g) = \chi(hgh^{-1}) \quad \text{for all } g, h \in G.$$

Definition 1.5 (Irreducibility). A representation $\rho : G \to \mathrm{GL}(V)$ is called *irreducible* if there is no proper, nonzero G-invariant subspace $W \subset V$.

Proposition 1.6. Every finite-dimensional representation of a finite group over $\mathbb{C}$ decomposes as a direct sum of irreducible representations.

Definition 1.7 (Group Algebra). Let F be a field. The group algebra $F[G]$ is the set of formal sums

$$\sum_{g \in G} a_g g, \quad a_g \in F,$$

with multiplication induced by the group operation.

Theorem 1.8 (Maschke's Theorem). Let F be a field such that $\mathrm{char}(F) \nmid |G|$. Then every representation of G over F is completely reducible.

Definition 1.9 (Inner Product of Characters). Given two complex characters χ, ψ, define:

$$\langle \chi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

Theorem 1.10 (Orthogonality of Irreducible Characters). Let $\chi, \psi \in \mathrm{Irr}(G)$. Then

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } \chi = \psi, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 1.11. The number of irreducible representations of G over $\mathbb{C}$ equals the number of conjugacy classes in G.

2. Examples

Example 1.12 (Trivial Representation). Let $V = \mathbb{C}$, and $\rho(g) = 1$ for all $g \in G$. Then $\chi(g) = 1$, and it is irreducible.

Example 1.13 (Regular Representation). Let $V = \mathbb{C}[G]$, and $\rho(g)(h) = gh$. This decomposes as a direct sum of all irreducible representations, each appearing with multiplicity equal to its degree.

Example 1.14. Compute the character table of S_3, and verify orthogonality of irreducible characters.

3. Counterexamples

Counterexample 1.15 (Failure of Maschke's Theorem). Let $G = \mathbb{Z}/2\mathbb{Z}$, and $F = \mathbb{F}_2$. Then $\mathrm{char}(F) \mid |G|$, and Maschke's theorem fails: the group algebra $F[G]$ is not semisimple, and not all representations are completely reducible.

4. Summary

This lecture sets the groundwork for understanding:

- Decomposition of representations
- The role of conjugacy classes
- The logic behind character tables
- When and why semisimplicity holds or fails (paving the way to modular representations)

Coming Up in Lecture 2: We'll explore *modular representation theory*, define *Brauer characters*, and study how ordinary character theory changes when the characteristic divides the group order.